G grupa, $H \leq G$ ježbe VII

$(\forall x,y \in G)$ $\quad x \sim y \Leftrightarrow xy^{-1} \in H$

$(ab)^{-1} = b^{-1}a^{-1}$

$ab(b^{-1}a^{-1}) = e$

1) $x \sim x \Leftrightarrow xx^{-1} = e \in H$

2) $x \sim y \Leftrightarrow xy^{-1} \in H$

$$\Downarrow$$

$$(xy^{-1})^{-1} = yx^{-1} \in H$$

$$\Downarrow$$

$$y \sim x$$

3) $x \sim y \wedge y \sim z$

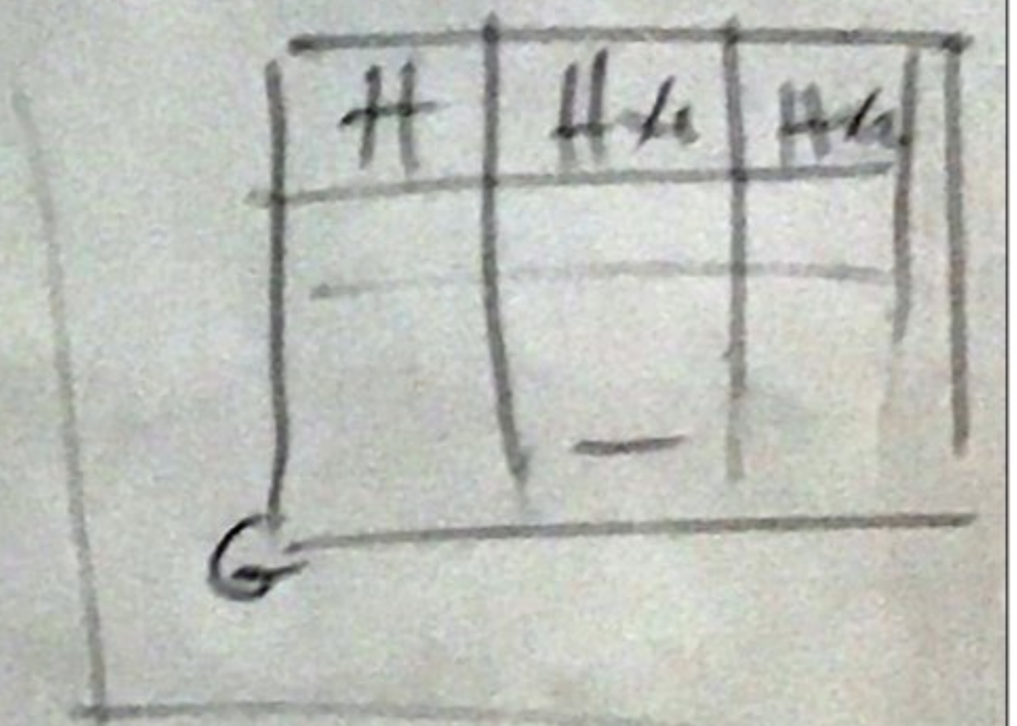
$xy^{-1} \in H \wedge yz^{-1} \in H$

$xy^{-1} \cdot y \cdot z^{-1} = xz^{-1} \in H$

$x \sim z$

$C_x = \{a \in G \mid a \sim x\} = \{a \in G \mid ax^{-1} \in H\} = \{a \in G \mid \exists h \in H \ ax^{-1} = h\} = \{a \in G \mid \exists h \in H \ a = hx\}$

$= Hx$ desni koset

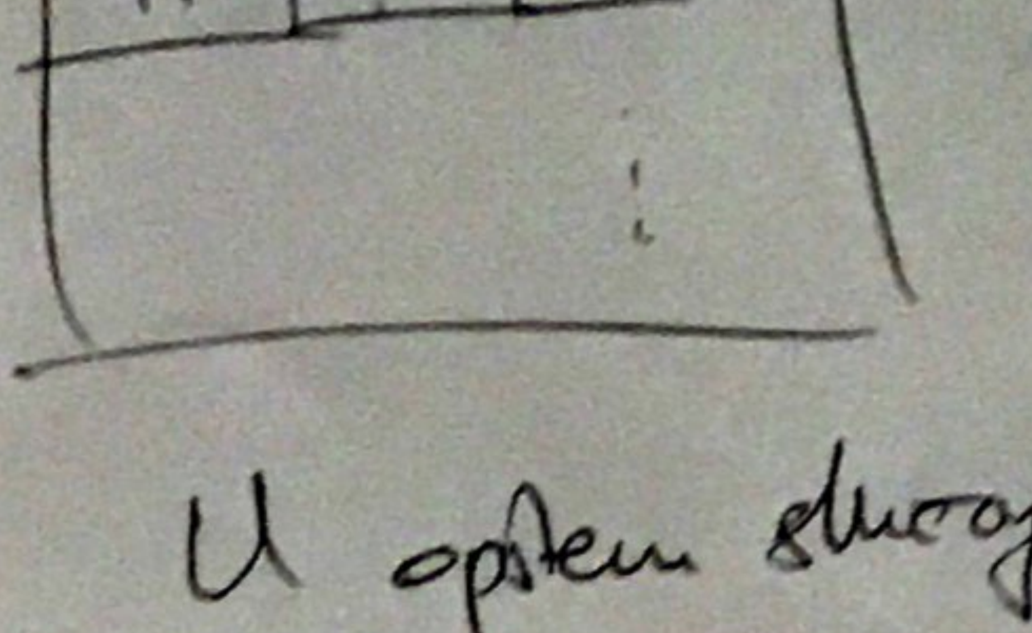


Analogno:

$\forall x,y \in G$ $\quad x \,S\, y \Leftrightarrow x^{-1}y \in H$

1), 2), 3)

$C_x = xH$ $\quad$ lijevi koset



U općem slučaju $\quad xH \neq Hx$

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Zad

$f_1(x) = x$ $\quad f_4(x) = \frac{x-1}{x}$ $\quad f_i$. preslikavanja

$f_2(x) = \frac{1}{x}$ $\quad f_5(x) = \frac{x}{x-1}$

$f_3(x) = 1-x$ $\quad f_6(x) = \frac{1}{1-x}$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_6	f_5	f_4	f_3
f_3	f_3	f_4	f_1	f_2	f_6	f_5
f_4	f_4	f_3	f_5	f_6	f_2	f_1
f_5	f_5	f_6	f_4	f_3	f_1	f_2
f_6	f_6	f_5	f_2	f_1	f_3	f_4

G je grupa sa operacijom $\circ$

$$f_5 \circ f_6(x) = f_5\left(\frac{1}{1-x}\right) = \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{\frac{1}{1-x}}{\frac{1-1+x}{1-x}} = \frac{1}{x} = f_2$$

$H = \{f_1, f_2\} \leq G$

$f_3 \circ H = \{f_3 \circ f_1, f_3 \circ f_2\} = \{f_3, f_4\}$

$\neq$

$H \circ f_3 = \{f_1 \circ f_3, f_2 \circ f_3\} = \{f_3, f_6\}$

Def Za podgrupu $H \leq G$ kažemo da je normalna podgrupa ako $\forall x \in G$ $\quad xH = Hx$

$\Longleftrightarrow$ 1) $\forall x \in G \quad xHx^{-1} = H$

2) $\forall x \in G \quad xHx^{-1} \subseteq H$

3) $\forall x \in G \quad \forall h \in H \quad xhx^{-1} \in H$

Primjer $H = \{e\} \trianglelefteq G$

$H = G \trianglelefteq G$

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Komutator:

$x, y \in G$

$[x, y] = x y x^{-1} y^{-1}$ - komutator el. x, y

Najmanja podgrupa grupe G koja sadrži sve komutatore grupe G zove se komutant / izvodna grupa.

Oznake:

$H \trianglelefteq G$ - normalna podgrupa

(1.) Neka je G Abelova grupa. Tada je svaka njena podgrupa normalna podgrupa.

$H \leq G$

Treba dokazati da je $H \trianglelefteq G$.

$x \in G$ - proizvoljno

$h \in H$ - proizvoljno

? $x h x^{-1} \in H$?

Trivijalno: $x h x^{-1} = h \underbrace{x x^{-1}}_{e} = h e = h \in H$

(2.) Ako su H i K normalne podgrupe grupe G, tada je i $H \cap K$ normalna podgrupa grupe G.

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$H \cap K \leq G$?

$x, y \in H \cap K$

Treba dokazati da je $xy^{-1} \in H \cap K$

$$\left.\begin{array}{l} xy^{-1} \in H \ (x, y \in H,\ H \leq G) \\ xy^{-1} \in K \ (x, y \in K,\ K \leq G) \end{array}\right\} \Rightarrow$$

$\Rightarrow H \cap K \leq G$

$h \in H \cap K$

$x \in G$

$xhx^{-1} \overset{?}{\in} H \cap K$

$$\left.\begin{array}{l} xhx^{-1} \in H \ (h \in H \cap K \Rightarrow h \in H,\ H \trianglelefteq G) \\ xhx^{-1} \in K \ (h \in H \cap K \Rightarrow h \in K,\ K \trianglelefteq G) \end{array}\right\} \Rightarrow$$

$\Rightarrow xhx^{-1} \in H \cap K \Rightarrow H \cap K \trianglelefteq G$

3. Dokazati da je u grupi G svaka podgrupa H, koja sadrži komutant G (grupa G') normalna podgrupa grupe G.

$[x, y] = xyx^{-1}y^{-1}, \quad x, y \in G$

$G' \leq H$ $\qquad$ $h \in H$

$H \trianglelefteq G$? $\qquad$ $x \in G$

$? \ xhx^{-1} \in H \ ?$

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$$\underbrace{xhx^{-1}h^{-1}}_{\substack{[x,h] \\ \cap \\ G' \\ \cap \\ H}} \cdot \underbrace{h}_{\in H} \in H$$

(4) Neka je G Abelova grupa i $H \subseteq G$ koja se sastoji od jediničnog elementa i svih elemenata reda 2 u grupi G. Dokazati da je $H \trianglelefteq G$.

<u>Red elementa</u>

$x \in G$

$ord(x)$ – najmanji prirodan broj n takav da je $x^n = e$ tj. $\underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ puta}} = e$.

$x^1 = e \Rightarrow x = e$

$ord(e) = 1$

$$H = \{e\} \cup \{x \in G \mid ord(x) = 2\} = \{x \in G \mid x^2 = e\}$$

$x, y \in H$

$? \; xy^{-1} \in H \,? \iff (xy^{-1})^2 = e \,?$

$$y \in H \left\langle \begin{array}{l} y = e \\ \\ y^2 = e \end{array} \right\} \iff \underline{\underline{y^2 = e}}$$

$y^2 = e$

$y \cdot y = e \,/\cdot y^{-1}$

$y \cdot \underbrace{(y y^{-1})}_{e} = e y^{-1}$

$$y = y^{-1}$$

$(x y^{-1})^2 = e\,?$

$(x y^{-1})^2 = (x y)^2 = (x y)\cdot(x y) \overset{\text{Abelova grupa}}{=} x \underbrace{y \cdot y}_{e} x = x^2 = e$

(5.) Neka je G grupa konačnog reda (konačna grupa). Tada svaki element iz G ima konačan red.

Posmatrajmo skup

$H = \{x, x^2, \dots, x^n, x^{n+1}\}$, gdje je $n = |G|$

$H \subseteq G$

$\exists k, l$ t.d. $x^k = x^l$

Uzmimo $k > l$.

$\underbrace{x \cdots x}_{k \text{ puta}} = \underbrace{x \cdots x}_{l \text{ puta}} \,/\, x^{-1}$ l puta

$\underbrace{x \cdots x}_{k-l \text{ puta}} = e \Rightarrow x^{k-l} = e$

$\Downarrow$

$(\exists n \in \mathbb{N})\ x^n = e \Rightarrow \operatorname{ord}(x) < \infty$

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6. Dokazati da u grupi mogu da postoje elementi konačnog reda, čiji je proizvod beskonačnog reda.

$GL(2, \mathbb{R})$

$\downarrow$

grupa reg. mat. reda 2×2 sa elementima iz $\mathbb{R}$

$\forall A \in GL(2, \mathbb{R}) \quad \det A \neq 0$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

$\underline{\text{ord}(A) = 2}$

$$B^2 = B \cdot B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

$\underline{\text{ord } B = 2}$

$$A \cdot B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$(AB)^2 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

$$(AB)^3 = (AB^2)(AB) = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix}$$

$$(AB)^n = (-1)^n \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix} \quad \text{(dokaz indukcijom)}$$

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$(AB)^n \neq E, \ \forall n \in \mathbb{N} \Rightarrow AB$ je beskonačnog reda ?.

(7.) a) $Z(G) \trianglelefteq G$

b) $H \leq Z(G) \Rightarrow H \trianglelefteq G$

c) $H \trianglelefteq G \Rightarrow Z(H) \trianglelefteq G$

a) Već dokazano da je $Z(G) \leq G$. Treba dokazati da je $Z(G) \trianglelefteq G$.

$$Z(G) = \{ g \in G \mid gx = xg, \ \forall x \in G \}$$

$g \in Z(G)$

$x \in G$

$xgx^{-1} \in Z(G)$?

$(xgx^{-1})a \overset{?}{=} a(xgx^{-1})$

$\boxed{\text{Osnovna teorema o homomorfizmima}}$

$$(xgx^{-1})a = (g\underbrace{xx^{-1}}_{e})a = ga \underset{\text{jer je } g \in Z(G)}{=} ag = axx^{-1}g =$$

$$= a(xgx^{-1}) \Rightarrow xgx^{-1} \in Z(G)$$

$$\Downarrow$$

$$Z(G) \trianglelefteq G$$

b) $\boxed{A \leq B, \ B \leq G \Rightarrow A \leq G}$

$H \leq Z(G) \wedge Z(G) \leq G \Rightarrow H \leq G$

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$h \in H \qquad x h x^{-1} \overset{?}{\in} H$

$x \in G$

$x h x^{-1} = h x x^{-1} = h \in H \Rightarrow H \trianglelefteq G$

c) $H \trianglelefteq G \overset{?}{\Rightarrow} Z(H) \trianglelefteq G$

$z \in Z(H) \qquad Z(H) = \{ h \in H \mid \forall z \in H \ hz = zh \}$

$g \in G$

$g z g^{-1} \overset{?}{\in} Z(H)$

$h \in H$

$(g z g^{-1}) h \overset{?}{=} h (g z g^{-1})$

$$\underline{h(g z g^{-1})} = \underbrace{g g^{-1} h g}_{\in H} z g^{-1} = g z (g^{-1} h g) g^{-1} =$$

già che $H \trianglelefteq G$

$$= g z g^{-1} h = \underline{(g z g^{-1}) h}$$

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9. Neka je $G = (\mathbb{R}^2, +)$ grupa čiji su elementi tačke u ravni $\mathbb{R} \times \mathbb{R}$, a operacija $+$ definisana po koordinatama. Dokazati da je:

$$L = \left\{ (x, mx) \mid x \in \mathbb{R} \right\}$$ podgrupa grupe G.

fiksirano

Odrediti klase (lijeve) grupe G i dati geometrijsko tumačenje.

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$(\mathbb{R}^2, +)$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$e = (0, 0)$

$(a, b \in L)$? $a - b \in L$?

$a = (x, mx)$

$b = (y, my)$

$a - b = (x, mx) + (-y, -my) =$

$= (x - y, mx - my) = (x - y, m(x - y)) \in L$

$\Rightarrow L \leq G$

$(a, b) \in \mathbb{R}^2$, m - koef.

$y - b = m(x - a) = l(m, (a, b))$

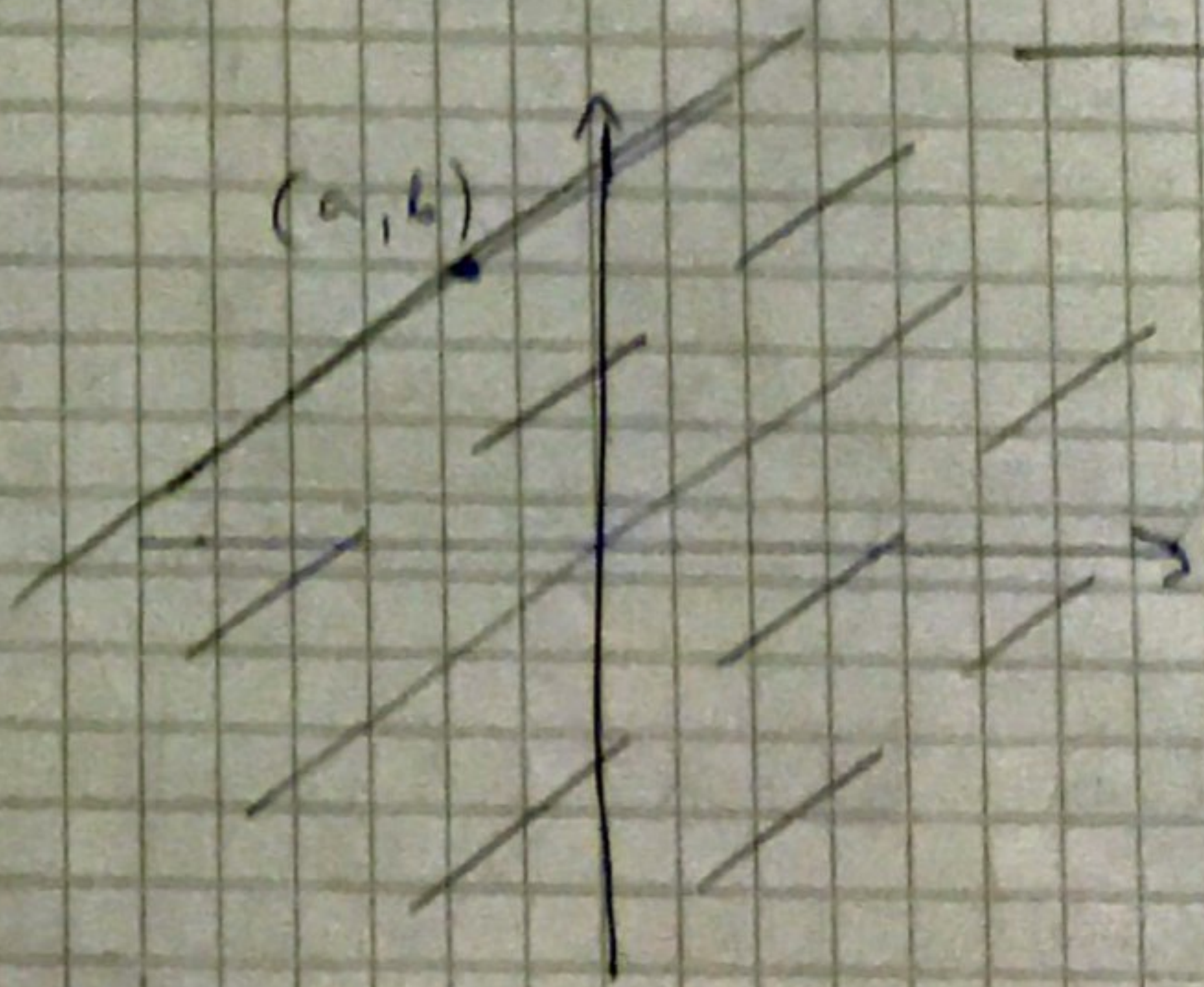
pravac koja prolazi kroz (a, b) i ima koef. pr. $\underline{m}$

$(a, b) \sim (x, y)$

$(x - a, y - b) \in L$

$\Downarrow$

$y - b = m(x - a)$



(Zad) H podgrupa indeksa 2, tada je $H \trianglelefteq G$.

(R) $G = H \cup x_0H$, $x_0 \notin H$.

Treba dok. da je $xH = Hx$, $\forall x \in G$.

1) $x \in H$

$\underline{xH = H}$

$\underline{xH \subseteq H}$

$h \in H$

$\underbrace{x}_{\in H}\underbrace{h}_{\in H} \in H$

$\underline{H \subseteq xH}$

$h \in H$

$h = x\underbrace{x^{-1}h}_{\in H} \in xH$

Analogno, $\underline{Hx = H}$.

$xH = Hx$

2). $x \notin H$.

tada $xH \neq H$ inače $x \cdot e = x \in H$

$\Rightarrow xH = \underline{G \setminus H} = x_0H$

Analogno $Hx = \underline{G \setminus H}$

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